EXAM
Probabilité Avancée
Travail Ecrit Autumn Semester 2015
January 11 2016

Length of exam : 3h minutes

Permitted : no pages of personnel notes are allowed.

No calculation machines are allowed.
Please start each question on a fresh page.

Attempt all the questions

Any dishonesty will be severely sanctioned!

Write first your full name and section :

Name :______________________________ **Given name :** ______________________________

Departement : _____________________

Exercice	**Points**
1	
2	
3	
4	
5	
Total points :	

1i) Define convergence in distribution for a sequence of random variables X_n and limit random variable X. Give two equivalent (and clearly distinct) equivalent conditions.
(ii) If for distribution functions F_n and F, we have that $F_n(t) \to F(t)\ \forall t \in D$ a dense subset of $\mathbb{R}$, show that $F_n \xrightarrow{D} F$.
(iii) What does it mean for probability measure μ to be the *law* of random variable X? If X_i are i.i.d. random variables with law μ and $\mu_n(\omega)$ s the random probability measure

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}.$$

Show that with probability one $\mu_n(\omega) \xrightarrow{D} \mu$.

2a) Let $X_1, X_2, \ldots,$ be independant identically distributed r.v.s taking values in $\mathbb{N} \cup -1 \backslash \{0, 1\}$, so that

$$\forall n \geq 2 \;\; P(X_1 = n) = \frac{C}{n^2 log^2(n)},$$

$$P(X_1 = -1) = 1 - \sum_n \frac{C}{n^2 log^2(n)};$$

where C is chosen so that $E(X_1) = 0$. We define $S_n = \sum_1^n X_i$. Show that

$$\frac{S_n}{n} \;\rightarrow\; 0,$$

in probability. Is it possible to extend this to a.s. convergence? Using the truncation at level $\frac{n}{log^{3/2}(n)}$ or otherwise, show that

$$\frac{S_n}{n/log(n)} \;\rightarrow\; -C,$$

in probability. Is it possible to extend this to convergence a.s.? Justify. (You may use approximations of sums via corresponding integrals without proof.)

3i) Suppose that $X_n \stackrel{D}{\to} X$ and $Y_n \stackrel{D}{\to} Y$. Does $X_n + Y_n \stackrel{D}{\to} X + Y$? Repeat the question with convergence in distribution replaced by convergence in probability.

(ii) For a metric space (E, d) we say that $X_n \stackrel{D}{\to} X$ for random elements in (E, d) if

$$E(g(X_n)) \to E(g(X))$$

for each bounded continuous function g on (E, d). Show that this implies that for every open set $O \subset E$

$$\liminf P(X_n \in O) \ \geq \ P(X \in O).$$

4) Let $\mathcal{H}$ be the set of bounded continuous functions on $\mathbb{R}$. For any compact interval $I \subset \mathbb{R}$ of nonzero length, define

$$\phi_I : \mathcal{H} \ \to \mathbb{R}$$

by

$$\phi_I f \ = \ \sup_{y \in I} f(x).$$

a) Let $\mathcal{G}$ be the $\sigma-$field of subsets of $\mathcal{H}$ generated by the applications of the form ϕ_I. What does this mean ? Show that this is equipvalent to a simpler $\sigma-$field.

b) Let $g : \mathbb{R} \to \mathbb{R}$ be in $\mathcal{H}$. Define $F : (\mathbb{R}, \mathcal{B}) \ \to \ (\mathcal{H}, \mathcal{G})$ by

$$F(x)(y) \ = \ \sup_{z \in [y - |x|/2, y + |x|/2]} g(z).$$

Is F a mesurable function. Justify.

c) For random variables $X, X_n n = 1, 2, \cdots$ define intervals

$$I_n = [X_n - 1, X_n + 1], \quad I \ = \ [X - 1, X + 1].$$

What does ϕ_{I_n} converges point wise to ϕ_I a.s. mean ? What does it imply for the given random variables. Can you find some $f \in \mathcal{H}$ so that $\phi_{I_n} f \ \to \phi_{[-1,1]} f$ implies X_n converges a.s. to zero. is this true for all $f \in \mathcal{H}$.

5) (i) Is the collection of subsets of $\mathbb{R}^2$ of the form

$$\{(x, y) : xa_1 + ya_2 \leq c\}$$

a π-system ? Justify.
(ii) For X_n i.i.d. $U(0, 1)$ random variables what are

$$\liminf_n n^{1/3} \max \{X_n, X_{n+1}, X_{n+2}\}$$

and

$$\liminf_n log(n) n^{1/3} \max \{X_n, X_{n+1}, X_{n+2}\}.$$

(iii)Let X_n be independent random variables with X_n having distribution $N((-1)^n \frac{1}{n}, \frac{1}{n})$. Does $S_n = \sum_{I=1}^{n} X_i$ converge a.s. as n tends to infinity. Change the parameters to get X_n converging on probability but the answer is opposite to the first answer. The smaller the change the greater the points.